

A family of linear codes with three weights[☆]

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Abstract

Recently, linear codes constructed by defining sets have attracted a lot of study, and many optimal linear codes with a few weights have been produced. The objective of this paper is to present a class of binary linear codes with three weights.

Keywords: linear codes, weight distributions, exponential sums

1. Introduction

Throughout this paper, $q = 2^m$ for a positive integer $m = 2e$, and h denotes a proper divisor of m . Let \mathbb{F}_q and \mathbb{F}_q^* denote the finite field with q elements and the multiplicative group of \mathbb{F}_q , respectively. Let \mathbb{F}_2^n denote the vector space of all n -tuples over \mathbb{F}_2 .

For $\mathbf{x} \in \mathbb{F}_2^n$, the (Hamming) weight $\text{wt}(\mathbf{x})$ is referred to the number of nonzero coordinate in \mathbf{x} . The (Hamming) distance $d(\mathbf{x}, \mathbf{y})$ between vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$ is defined to be the weight $\text{wt}(\mathbf{x} - \mathbf{y})$ of $\mathbf{x} - \mathbf{y} \in \mathbb{F}_2^n$. The minimum distance of \mathcal{C} is the least Hamming distance between two distinct codewords. If \mathcal{C} is a k -dimensional subspace of \mathbb{F}_2^n with minimum distance d , then \mathcal{C} will be called an $[n, k, d]$ linear code.

Let A_i , also denoted by $A_i(\mathcal{C})$, indicate the number of codewords of weight i in \mathcal{C} . The list A_i ($0 \leq i \leq n$) is called the weight distribution of \mathcal{C} . This

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distribution is usually recorded as the coefficients of a polynomial, the weight enumerator, which is defined as

$$W_{\mathcal{C}}(x) = \sum_{\mathbf{c} \in \mathcal{C}} x^{\text{wt}(\mathbf{c})} = \sum_{i=0}^n A_i x^i.$$

Clearly, the weight distribution derives the minimum distance, and thus the error correction capability. In this connection, it would be useful in determining as much as one can about the weight distribution of a specific code or a family of a code [1, 3, 4, 17, 18, 20]. If the number of nonzero A_i in the sequence (A_1, A_2, \dots, A_n) equals t , then \mathcal{C} is called a t -weight code.

Let $D = \{d_1, d_2, \dots, d_n\} \subseteq \mathbb{F}_q$ and Tr denote the trace function from \mathbb{F}_q onto \mathbb{F}_2 . Then a binary linear code associated with D can be defined as

$$\mathcal{C}_D = \{(\text{Tr}(xd_1), \text{Tr}(xd_2), \dots, \text{Tr}(xd_n)) : x \in \mathbb{F}_q\}. \quad (1.1)$$

The set D is said to be the defining set of \mathcal{C}_D . This construction of linear codes was proposed by Ding [5, 6, 11, 12] and has attracted a lot of attention in the last nine years. This approach is generic as many classes of constant, 2-weight and 3-weight linear codes were produced [7, 8, 9, 13, 16, 22, 23].

For $a \in \mathbb{F}_2$, the defining set D_a is given by

$$D_a = \left\{ x \in \mathbb{F}_q^* : \text{Tr} \left(x^{2^h+1} + x \right) = a \right\}.$$

Obviously, if $a = 0$ and $h = 1$, then

$$D_0 = \left\{ x \in \mathbb{F}_q^* : \text{Tr} \left(x^3 + x \right) = 0 \right\},$$

that is the same as the defining set of \mathcal{C}_D depicted in [19]. Thus the code \mathcal{C}_{D_a} presented in this paper is a generalization of that in [19].

In this paper, we will study the weight distribution of \mathcal{C}_{D_a} of (1.1). As it turns out that \mathcal{C}_a is a linear code with 3 weights, if $m/h > 2$. This implies that this kind of linear codes may be of use in secret sharing schemes [13]. We should mention that the idea of solving the weight distribution of \mathcal{C}_{D_a} is inspired by that in [19], which was well-written and inspiring.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and basic results of exponential sums over finite fields, and present some useful conclusions in [2]. Section 3 is devoted to investigating the weight distribution of the linear codes defined in (1.1). In Section 4, we conclude this paper, and make some comments on the applications of these codes in secret sharing schemes.

2. Preliminaries

In this section, we firstly introduce the definition of group characters. Then we present some lemmas, which are necessary in proving the main results.

Let G be a finite abelian group. A *character* χ of G is a homomorphism from G into the multiplicative group U of complex numbers of absolute value 1—that is, a mapping from G into U with

$$\chi(g_1 g_2) = \chi(g_1) \chi(g_2)$$

for all $g_1, g_2 \in G$ [15]. The character χ_0 defined by $\chi_0(g) = 1$ for all $g \in G$ is called the *trivial* character of G . All other characters of G are called *nontrivial*.

The following lemma present a basic property of characters.

Lemma 1 ([15], Theorem 5.4). *If χ is a nontrivial character of the finite abelian group, then*

$$\sum_{g \in G} \chi(g) = 0.$$

In a finite field there are two finite abelian groups—namely, the additive group and the multiplicative group of \mathbb{F}_q . Characters of the *additive group* are called *additive characters* of \mathbb{F}_q .

To present additive characters of a finite field, we need to introduce an important mapping from \mathbb{F}_{2^a} to \mathbb{F}_{2^b} , where a and b denote two integers and satisfy $b \mid a$.

Definition 2 ([15], Definition 2.22). *For $\alpha \in \mathbb{F}_{2^a}$, the trace $\text{Tr}_b^a(\alpha)$ of α over \mathbb{F}_{2^b} is defined by*

$$\text{Tr}_b^a(\alpha) = \alpha + \alpha^{2^b} + \cdots + \alpha^{2^{b(l-1)}},$$

where $l = a/b$. If $b = 1$, then $\text{Tr}_1^a(\alpha)$ is called the *absolute trace* of α .

For simplicity, we use Tr to denote the trace function from \mathbb{F}_q to \mathbb{F}_2 in this paper. By Theorem 2.23 in [15], $\text{Tr}_b^a(\alpha + \beta) = \text{Tr}_b^a(\alpha) + \text{Tr}_b^a(\beta)$ for all $\alpha, \beta \in \mathbb{F}_{2^a}$, and Tr_b^a is a linear transformation from \mathbb{F}_{2^a} onto \mathbb{F}_{2^b} , where both \mathbb{F}_{2^a} and \mathbb{F}_{2^b} are viewed as vector space over \mathbb{F}_{2^b} .

By the linearity of Tr , the function χ_1 defined by

$$\chi_1(c) = e^{2\pi i \text{Tr}(c)/2} = e^{\pi i \text{Tr}(c)} \text{ for all } c \in \mathbb{F}_q$$

is an additive character of \mathbb{F}_q .

For $a, b \in \mathbb{F}_q$, define the following exponential sum

$$S_h(a, b) = \sum_{x \in \mathbb{F}_q} \chi_1(x^{2^h+1} + bx),$$

where h is a proper divisor of m .

In order to determine the weight distribution of the code \mathcal{C}_{D_a} defined in (1.1), we need the values of $S_h(a, b)$, which are presented in the following lemmas.

Lemma 3 ([2], Theorem 4.1). *When m/h is odd, we have*

$$\sum_{x \in \mathbb{F}_q} \chi_1 \left(ax^{2^h+1} \right) = 0$$

for each $a \in \mathbb{F}_q^$.*

Lemma 4 ([2], Theorem 4.6). *Suppose m/h is odd, then we have*

$$S_h(1, 1) = \left(\frac{2}{m/h} \right)^h 2^{\frac{m+h}{2}},$$

where and hereafter (\cdot) denotes the Jacobi symbol.

Lemma 5 ([2], Theorem 4.2). *Let $b \in \mathbb{F}_q^*$ and suppose m/h is odd. Then*

$$S_h(a, b) = S_h(1, bc^{-1}),$$

where c is the unique element satisfying $c^{2^h+1} = a$. Further we have

$$S_h(1, b) = \begin{cases} 0, & \text{if } \text{Tr}_h^m(b) \neq 1, \\ \pm 2^{\frac{m+h}{2}}, & \text{if } \text{Tr}_h^m(b) = 1, \end{cases}$$

Lemma 6 ([2], Theorem 5.2). *Let m/h be even so that $m = 2e$ for some integer e . Then*

$$S_h(a, 0) = \begin{cases} (-1)^{\frac{e}{h}} 2^e, & \text{if } a \neq g^{t(2^h+1)} \text{ for any integer } t, \\ -(-1)^{\frac{e}{h}} 2^{e+h}, & \text{if } a = g^{t(2^h+1)} \text{ for some integer } t, \end{cases}$$

where g is a generator of \mathbb{F}_q^ .*

Lemma 7 ([2], Theorem 5.3). *Let $b \in \mathbb{F}_q^*$ and suppose m/h is even so that $m = 2e$ for some integer e . Let $f(x) = a^{2^h} x^{2^{2h}} + ax \in \mathbb{F}_q[x]$. There are two cases.*

1. *If $a \neq g^{t(2^h+1)}$ for any integer t , then f is a permutation polynomial of \mathbb{F}_q . Let x_0 be the unique element satisfying $f(x) = b^{2^h}$. Then*

$$S_h(a, b) = (-1)^{\frac{e}{h}} 2^e \chi_1 \left(ax_0^{2^h+1} \right).$$

2. *If $a = g^{t(2^h+1)}$ for some integer t , then $S_h(a, b) = 0$ unless the equation $f(x) = b^{2^h}$ is solvable. If this equation is solvable, with solution x_0 say, then*

$$S_h(a, b) = \begin{cases} -(-1)^{\frac{e}{h}} 2^{e+h} \chi_1 \left(ax_0^{2^h+1} \right), & \text{if } \text{Tr}_h^m(a) = 0, \\ (-1)^{\frac{e}{h}} 2^e \chi_1 \left(ax_0^{2^h+1} \right), & \text{if } \text{Tr}_h^m(a) \neq 0. \end{cases}$$

Lemma 8. *Let symbols be the same as before. Then the equation*

$$x^{2^{2h}} + x = 1 \quad (2.1)$$

has solutions in \mathbb{F}_q if and only if $m/h \equiv 0 \pmod{4}$.

Proof. Sufficiency: Suppose $m/h \equiv 0 \pmod{4}$. Note that

$$x^{2^{2h}} + x = \text{Tr}_{2h}^{4h}(x) \text{ for all } x \in \mathbb{F}_{2^{4h}}.$$

The equation (2.1) has solutions in $\mathbb{F}_{2^{4h}}$, as Tr_{2h}^{4h} is surjective from $\mathbb{F}_{2^{4h}}$ to $\mathbb{F}_{2^{2h}}$. Then the desired result follows from the fact that $\mathbb{F}_{2^{2h}} \subseteq \mathbb{F}_{2^m}$.

Necessity: Suppose $x_0 \in \mathbb{F}_q$ is a solution of (2.1). If $m/h \not\equiv 0 \pmod{4}$, then $m/h \equiv 2 \pmod{4}$ or $m/h \equiv 1 \pmod{2}$. For the case $m/h \equiv 2 \pmod{4}$, we have

$$\text{Tr}_{2h}^m(x_0^{2^{2h}} + x_0) = 2\text{Tr}_{2h}^m(x_0) = 0. \quad (2.2)$$

On the other hand, we obtain

$$\text{Tr}_{2h}^m(x_0^{2^{2h}} + x_0) = \text{Tr}_{2h}^m(1) = m/(2h) = 1,$$

which contradicts with (2.2).

If $m/h \equiv 1 \pmod{2}$, we get $(2m)/h \equiv 2 \pmod{4}$. By the discussion above, we know that the equation $x^{2^{2h}} + x = 1$ has no solution in $\mathbb{F}_{2^{2m}}$. So does it in $\mathbb{F}_q = \mathbb{F}_{2^m}$, since $\mathbb{F}_q \subseteq \mathbb{F}_{2^{2m}}$. \square

Lemma 9. *Let $m/h \geq 2$ be even. Then*

$$S_h(1, 1) = \begin{cases} 0, & \text{if } m/h \equiv 2 \pmod{4}, \\ -(-1)^{\frac{m}{4}} 2^{e+h}, & \text{if } m/h \equiv 0 \pmod{4}. \end{cases}$$

Proof. Note that $\text{Tr}_h^m(1) = 0$, as m/h is even. It then follows from Lemmas 7 and 8 that

$$S_h(1, 1) = \begin{cases} 0, & \text{if } m/h \equiv 2 \pmod{4}, \\ -2^{e+h} \chi_0(x_0^{2^h+1}), & \text{if } m/h \equiv 0 \pmod{4}, \end{cases}$$

where x_0 is a solution of the equation $x^{2^{2h}} + x = 1$, when $m/h \equiv 0 \pmod{4}$. By the proof of Lemma 8, we know that the solutions of (2.1) are just the preimage of $\text{Tr}_{2h}^{4h}(1)$ and obviously they are in $\mathbb{F}_{2^{4h}}$. If $m/(4h) \equiv 0 \pmod{2}$, then $m/4 \equiv 0 \pmod{2}$ and

$$\text{Tr}(x_0^{2^h+1}) = \frac{m}{4h} \text{Tr}_1^{4h}(x_0^{2^h+1}) = 0, \text{ i.e., } (-1)^{\text{Tr}(x_0^{2^h+1})} = (-1)^{\frac{m}{4}}.$$

If $m/(4h) \equiv 1 \pmod{2}$, then $m/4 \equiv h \pmod{2}$ and

$$\begin{aligned}\mathrm{Tr}\left(x_0^{2^h+1}\right) &= \frac{m}{4h}\mathrm{Tr}_1^{4h}\left(x_0^{2^h+1}\right) \\ &= \mathrm{Tr}_1^{4h}\left(x_0^{2^h+1}\right) \\ &= \mathrm{Tr}_1^{2h}\left(\mathrm{Tr}_{2h}^{4h}\left(x_0^{2^h+1}\right)\right).\end{aligned}\tag{2.3}$$

By the definition of the trace function and x_0 , we have

$$\begin{aligned}\mathrm{Tr}_1^{2h}\left(\mathrm{Tr}_{2h}^{4h}\left(x_0^{2^h+1}\right)\right) &= \mathrm{Tr}_1^{2h}\left(x_0^{2^h+1} + x_0^{2^{2h}(2^h+1)}\right) \\ &= \mathrm{Tr}_1^{2h}\left(x_0^{2^h+1} + (1+x_0)^{2^h+1}\right) \\ &= \mathrm{Tr}_1^{2h}\left(x_0^{2^h} + x_0 + 1\right) \\ &= \mathrm{Tr}_1^h\left(\mathrm{Tr}_h^{2h}\left(x_0^{2^h} + x_0 + 1\right)\right) \\ &= \mathrm{Tr}_{h/1}(1) \\ &= h.\end{aligned}\tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\mathrm{Tr}\left(x_0^{2^h+1}\right) = h, \text{ i.e., } (-1)^{\mathrm{Tr}\left(x_0^{2^h+1}\right)} = (-1)^{\frac{m}{4}}.$$

Hence, we get

$$S_h(1, 1) = \begin{cases} 0, & \text{if } m/h \equiv 2 \pmod{4}, \\ -(-1)^{\frac{m}{4}}2^{e+h}, & \text{if } m/h \equiv 0 \pmod{4}. \end{cases}$$

This completes the proof of this lemma. \square

3. Results and Proofs

In this section, we will determine the parameters of the code \mathcal{C}_{D_a} ($a \in \mathbb{F}_2$) defined in (1.1), and give the proofs of these parameters.

Define

$$N_a = \left| \left\{ x \in \mathbb{F}_q : \mathrm{Tr}\left(x^{2^h+1} + x\right) = a \right\} \right|.$$

By definition, the length n_a of the code \mathcal{C}_{D_a} satisfies

$$n_a = N_a + a - 1.\tag{3.1}$$

It can be easily checked that

$$\begin{aligned}
N_a &= 2^{-1} \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_2} (-1)^{y \text{Tr}(x^{2^h+1}+x) - ya} \\
&= 2^{-1} \sum_{x \in \mathbb{F}_q} \left(1 + (-1)^{\text{Tr}(x^{2^h+1}+x) - a} \right) \\
&= 2^{m-1} + 2^{-1} \sum_{x \in \mathbb{F}_q} (-1)^{\text{Tr}(x^{2^h+1}+x) - a} \\
&= 2^{m-1} + 2^{-1} (-1)^a S_h(1, 1)
\end{aligned} \tag{3.2}$$

Define

$$N(a, b) = \left| \left\{ x \in \mathbb{F}_q : \text{Tr}(x^{2^h+1} + x) = a \text{ and } \text{Tr}(bx) = 0 \right\} \right|.$$

Denote $\text{wt}(\mathbf{c}_b)$ the Hamming weight of the codeword \mathbf{c}_b with $b \in \mathbb{F}_q^*$ of the code \mathcal{C}_{D_a} . It is easy to see that

$$\text{wt}(\mathbf{c}_b) = N_a - N(a, b). \tag{3.3}$$

For any $b \in \mathbb{F}_q^*$, by Lemma 1, we have

$$\begin{aligned}
N(a, b) &= 2^{-2} \sum_{x \in \mathbb{F}_q} \left(\sum_{y \in \mathbb{F}_2} (-1)^{y \text{Tr}(x^{2^h+1}+x) - ya} \right) \left(\sum_{z \in \mathbb{F}_2} (-1)^{z \text{Tr}(bx)} \right) \\
&= 2^{-2} \sum_{x \in \mathbb{F}_q} \left(1 + (-1)^{\text{Tr}(x^{2^h+1}+x) - a} \right) (1 + (-1)^{\text{Tr}(bx)}) \\
&= 2^{m-2} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\text{Tr}(x^{2^h+1}+x) - a} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\text{Tr}(x^{2^h+1}+bx+x) - a} \\
&= 2^{m-2} + 2^{-2} (-1)^a (S_h(1, 1) + S_h(1, b+1))
\end{aligned} \tag{3.4}$$

Theorem 10. *Let m/h be odd. For $a \in \mathbb{F}_2$, the code \mathcal{C}_{D_a} defined in (1.1) is a $[n_a, m]$ binary linear code with the weight distribution in Table 1, where $n_a = 2^{m-1} + (-1)^a \left(\frac{2}{m/h} \right)^h 2^{\frac{m+h-2}{2}} + a - 1$.*

Proof. We only prove the conclusions on the parameters of the code \mathcal{C}_{D_0} , since the conclusions on \mathcal{C}_{D_1} can be similarly proved.

By Lemma 9, (3.1) and (3.2), we obtain

$$n_0 = N_0 - 1 = 2^{m-1} + \left(\frac{2}{m/h} \right)^h 2^{\frac{m+h-2}{2}} - 1. \tag{3.5}$$

Table 1: The weight distribution of the codes of Theorem 10

Weight w	Multiplicity A
0	1
2^{m-2}	$2^{m-h-1} - 1 + \left(\frac{2}{m/h}\right)^h 2^{\frac{m-h-2}{2}}$
$2^{m-2} + (-1)^a \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-2}{2}}$	$2^{m-h-1} - \left(\frac{2}{m/h}\right)^h 2^{\frac{m-h-2}{2}}$
$2^{m-2} + (-1)^a \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-4}{2}}$	$2^m - 2^{m-h}$

For $b \in \mathbb{F}_q \setminus \{0, 1\}$, by Lemma 5, we have

$$S_h(1, b+1) = \begin{cases} 0, & \text{if } \text{Tr}(b+1) \neq 1, \\ \pm 2^{\frac{m+h}{2}}, & \text{if } \text{Tr}(b+1) = 1. \end{cases}$$

For $b = 1$, by Lemma 3, $S_h(1, b+1) = S_h(1, 0) = 0$. Then for $b \in \mathbb{F}_q^*$, we get

$$S_h(1, b+1) \in \{0, \pm 2^{\frac{m+h}{2}}\}.$$

It follows from (3.4) that

$$N(a, b) \in \left\{ \mu, \mu \pm 2^{\frac{m+h}{2}} \right\},$$

where $\mu = 2^{m-2} + 2^{-2} \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h}{2}}$. Hence, the weight $\text{wt}(\mathbf{c}_b)$ of the codeword \mathbf{c}_b satisfies

$$\begin{aligned} \text{wt}(\mathbf{c}_b) &= N_0 - N(0, b) \\ &\in \left\{ 2^{m-2}, 2^{m-2} + \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-2}{2}}, 2^{m-2} + \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-4}{2}} \right\}, \end{aligned}$$

and the code \mathcal{C}_{D_0} has all the three weights in the above set. The minimum distance of the dual code $\mathcal{C}_{D_0}^\perp$ of \mathcal{C}_{D_0} is at least 3, since $0 \notin D_0$ and the elements in D_0 are pairwise distinct. Define

$$\begin{aligned} \omega_1 &= 2^{m-2}, \\ \omega_2 &= 2^{m-2} + \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-2}{2}}, \\ \omega_3 &= 2^{m-2} + \left(\frac{2}{m/h}\right)^h 2^{\frac{m+h-4}{2}}. \end{aligned}$$

Next we will calculate the multiplicity A_{ω_i} of ω_i . By the computation above, the number of $b \in \mathbb{F}_q^*$ such that $\text{wt}(\mathbf{c}_b) = \omega_3$ is equal to

$$1 + |\{b \in \mathbb{F}_q \setminus \{0, 1\} : \text{Tr}_h^m(b+1) \neq 1\}|.$$

It is known that [15]

$$|\{b \in \mathbb{F}_q \setminus \{0, 1\} : \text{Tr}_h^m(b+1) = 1\}| = 2^{m-h} - 1.$$

Hence, $A_{\omega_3} = 2^m - 2^{m-h}$. By the first two Pless Power Moment ([14], P. 260), we get the following system of equations

$$\begin{cases} A_{\omega_1} + A_{\omega_2} + A_{\omega_3} = 2^m - 1, \\ \omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} = n_0 2^{m-1}, \end{cases} \quad (3.6)$$

where n_0 is given by (3.5). Solving the system of equations in (3.6) yields the weight distribution of \mathcal{C}_{D_0} . The dimension of \mathcal{C}_{D_0} is m , as $\text{wt}(\mathbf{c}_b) > 0$ for $b \in \mathbb{F}_q^*$. The proof of the parameters of \mathcal{C}_{D_0} is completed. \square

Table 2: The weight distribution of the codes of Theorem 11

Weight w	Multiplicity A
0	1
2^{m-2}	$2^m - 2^{m-2h} - 1$
$2^{m-2} - 2^{e+h-2}$	$2^{m-2h-1} + (-1)^a 2^{e-h-1}$
$2^{m-2} + 2^{e+h-2}$	$2^{m-2h-1} - (-1)^a 2^{e-h-1}$

Theorem 11. *Let $2 < m/h \equiv 2 \pmod{4}$. Then the code \mathcal{C}_{D_a} defined in (1.1) is a $[2^{m-1} + a - 1, m]$ binary linear code with the weight distribution in Table 2.*

Proof. Similarly, we only prove the results of \mathcal{C}_{D_0} .

It follows from Lemma 9, (3.1) and (3.2) that the length n_0 of \mathcal{C}_{D_0} is given by

$$n_0 = N_0 - 1 = 2^{m-1} - 1.$$

Note that $\text{Tr}_h^m(1) = 0$, as m/h is even. For $b \in \mathbb{F}_q^* \setminus \{1\}$, by Lemma 7, we know $S_h(1, b+1) \in \{0, \pm 2^{e+h}\}$. If $b = 1$, by Lemma 6 $S_h(1, 0) = -2^{e+h}$. Hence, for $b \in \mathbb{F}_q^*$, we have

$$S_h(1, b+1) \in \{0, \pm 2^{e+h}\}.$$

It then follows from Lemma 9 and (3.4) that

$$N(0, b) \in \{2^{m-2}, 2^{m-2} \pm 2^{e+h-2}\}$$

for $b \in \mathbb{F}_q^*$. Hence,

$$\text{wt}(\mathbf{c}_b) \in \{2^{m-2}, 2^{m-2} \pm 2^{e+h-2}\},$$

for all $b \in \mathbb{F}_q^*$.

As stated in the proof of Theorem 10, the minimum distance d^\perp of the dual code $\mathcal{C}_{D_0}^\perp$ of \mathcal{C}_{D_0} satisfies $d^\perp \geq 3$.

Table 3: The weight distribution of \mathcal{C}_{D_1} , when $m/h = 2$

Weight w	Multiplicity A
0	1
2^{m-1}	1
2^{m-2}	$2^m - 2$

Table 4: The weight distribution of the codes of Theorem 12

Weight w	Multiplicity A
0	1
$2^{m-2} - (-1)^{\frac{m+4a}{4}} 2^{e+h-2}$	$2^m - 2^{m-2h}$
2^{m-2}	$2^{m-2h-1} - (-1)^{\frac{m}{4}} 2^{e-h-1} - 1$
$2^{m-2} - (-1)^{\frac{m+4a}{4}} 2^{e+h-1}$	$2^{m-2h-1} + (-1)^{\frac{m}{4}} 2^{e-h-1}$

Let $\omega_1 = 2^{m-2}$, $\omega_2 = 2^{m-2} + 2^{e+h-2}$ and $\omega_3 = 2^{m-2} - 2^{e+h-2}$. The first three Pless Power Moments ([14], P. 260) give the following system of equations

$$\begin{cases} A_{\omega_1} + A_{\omega_2} + A_{\omega_3} = 2^m - 1, \\ \omega_1 A_{\omega_1} + \omega_2 A_{\omega_2} + \omega_3 A_{\omega_3} = 2^{2m-2}, \\ \omega_1^2 A_{\omega_1} + \omega_2^2 A_{\omega_2} + \omega_3^2 A_{\omega_3} = (2^{m-1} + 1) 2^{2m-3}. \end{cases}$$

Solving the above system of equations, we get the weight distribution of Table 2. \square

Remark:

1. If $m/h = 2$, then the code \mathcal{C}_{D_0} defined in (1.1) is a $[2^{m-1} - 1, m - 1, 2^{m-2}]$ constant binary linear code. And the code $\mathcal{C}_{D_0} \setminus \{0\}$ is an optimal constant-weight code with Johnson Bound I.
2. If $m/h = 2$, then the code \mathcal{C}_{D_1} is a $[2^{m-1}, m]$ binary linear code with the weight distribution in Table 3.

For the case $m/h = 2$, we omit the proofs of the parameters of \mathcal{C}_{D_a} ($a \in \mathbb{F}_2$) and the details are left to the reader.

Theorem 12. *Let $m/h \equiv 0 \pmod{4}$. Then the code \mathcal{C}_{D_a} defined in (1.1) is a $[2^{m-1} - (-1)^{\frac{m+4a}{4}} 2^{e+h-1} + a - 1, m]$ binary linear code with the weight distribution in Table 4.*

Proof. The proof of this theorem is analogous to that in Theorem 11 and will not be included here. \square

Example 1. *Let $(m, h) = (9, 3)$. For $a = 0$, the code \mathcal{C}_{D_0} has parameters $[223, 9, 96]$ and weight distribution*

$$1 + 36x^{96} + 448x^{112} + 27x^{128};$$

for $a = 1$, the code \mathcal{C}_{D_1} has parameters $[288, 9, 128]$ and weight distribution

$$1 + 27x^{128} + 448x^{144} + 36x^{160}.$$

Example 2. Let $(m, h) = (12, 2)$. For $a = 0$, the code \mathcal{C}_{D_0} has parameters $[2047, 12, 960]$ and weight distribution

$$1 + 136x^{960} + 3839x^{1024} + 120x^{1088};$$

for $a = 1$, the code \mathcal{C}_{D_1} has parameters $[2048, 12, 960]$ and weight distribution

$$1 + 120x^{960} + 3839x^{1024} + 136x^{1088}.$$

Example 3. Let $(m, h) = (8, 2)$. For $a = 0$, the code \mathcal{C}_{D_0} has parameters $[96, 8, 32]$ and weight distribution

$$1 + 10x^{32} + 240x^{48} + 5x^{64};$$

for $a = 1$, the code \mathcal{C}_{D_1} has parameters $[160, 8, 64]$ and weight distribution

$$1 + 5x^{64} + 240x^{80} + 10x^{96}.$$

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